

ON THE GROMOV HYPERBOLICITY OF STRONGLY PSEUDOCONVEX DOMAINS IN ALMOST COMPLEX MANIFOLDS

FLORIAN BERTRAND AND HERVÉ GAUSSIER

ABSTRACT. Let D be a strongly J -pseudoconvex domain, with a connected boundary, in an almost complex manifold (M, J) . We give a complete result on the equivalence between the Gromov hyperbolicity and the Kobayashi hyperbolicity (or equivalently the Brody hyperbolicity) of D .

INTRODUCTION

Complex Finsler geometry is an important branch of differential geometry, generalizing Hermitian geometry and carrying precious information on the geometry of the ambient complex manifold; its development goes back to the works of Carathéodory who introduced the Carathéodory pseudometric. The interest in complex Finsler geometry increased with the works of S.Kobayashi and his characterization of ample vector bundles, see [21]. We will focus in our paper on the Kobayashi metric, another well-known example of a complex Finsler pseudometric. One could refer to [22] for a presentation of complex hyperbolic spaces. When the integrated pseudodistance associated to that metric is a distance, the corresponding metric space is called Kobayashi hyperbolic and the complex manifold carrying the distance inherits complex dynamical properties particularly adapted to study spaces of holomorphic maps. If this is the case of any bounded domain in \mathbb{C}^n , curvature conditions on the boundary furnish information on the global behaviour of geodesics in the associated metric space, as in the celebrated paper of L.Lempert [23] for smooth strongly convex domains, or on their local boundary behaviour, as in [14, 17, 24] for strongly pseudoconvex domains.

Gromov hyperbolic spaces were introduced by M.Gromov [18] in the eighties as geodesic metric spaces in which geodesic triangles are thin. See for instance [7, 8, 11, 16] for different extensions of the theory. Although the two theories developed independently, the Kobayashi metric being restricted to complex geometry and the Gromov hyperbolicity having deep ramifications in group theory, it is quite natural to study their links in almost complex manifolds. The Poincaré half plane and as a generalization, the unit ball in \mathbb{C}^n , are famous examples of Kobayashi hyperbolic domains and of Gromov hyperbolic spaces. In Kobayashi hyperbolic spaces, under curvature conditions, real geodesics should behave quite similarly as in the Poincaré half space or in the unit ball of \mathbb{C}^n , and hence such metric spaces should enter the class of hyperbolic Gromov spaces. The first result in that direction was due to Z.Balogh and M.Bonk [2] for bounded strictly pseudoconvex domains in \mathbb{C}^n . Partial generalizations were obtained by F.Bertrand [4] and L.Blanc-Centi [6]. A different approach, linking the Kobayashi hyperbolicity and symplectic hyperbolicity, is proposed in [5].

The aim of the paper is to prove the following general result on the equivalence between the Gromov hyperbolicity and the Kobayashi hyperbolicity (or equivalently the Brody hyperbolicity) of some almost complex manifolds, under curvature conditions.

2000 *Mathematics Subject Classification.* 32Q45, 32Q60, 32T15, 58E05.

Key words and phrases. Almost complex manifold, Gromov hyperbolicity, Kobayashi hyperbolicity, Brody hyperbolicity, Morse theory.

Theorem 0.1. *Let D be a smooth relatively compact domain in an almost complex manifold (M, J) . We assume that the boundary ∂D of D is connected and that ∂D is strictly J -pseudoconvex. Then $(D, d_{(D, J)})$ is Gromov hyperbolic if and only if D does not contain any Brody J -holomorphic curve.*

We first point out that a product of two strictly pseudoconvex domains is not Gromov hyperbolic, according to [2]. That result explains the curvature condition, namely the strict pseudoconvexity, imposed on the domain in Theorem 0.1. We also point out that the strict J -pseudoconvexity of the boundary ∂D of D does not ensure the Kobayashi hyperbolicity of (D, J) . More precisely, one can prove that a relatively compact domain in an almost complex manifold (M, J) , strictly J -pseudoconvex, is complete hyperbolic in the sense of Kobayashi if and only if it does not contain any Brody J -holomorphic curve (see, for instance, [20, 9]). The following example presents a strict pseudoconvex domain, relatively compact in a complex projective space, that is not Kobayashi hyperbolic.

Example 0.2. *Consider a homogeneous complex polynomial P such that the set $\{P = 0\}$ is contained in \mathbb{CP}^n . One can smoothly deform the set $\{|P|^2 = 1\}$ into a strictly pseudoconvex hypersurface in \mathbb{CP}^n . That deformed hypersurface bounds a domain D in \mathbb{CP}^n containing an entire curve (in the set $\{P = 0\}$); D is not Kobayashi hyperbolic.*

Finally in contrast with the case of complex manifolds, D. McDuff constructed in [25] a relatively compact domain in an almost complex manifold, with a disconnected strictly J -pseudoconvex boundary. We do not consider such domains with non connected boundary in the paper, see Remark 2.4. It would be interesting to understand their geometry from a complex point of view.

As a corollary of Theorem 0.1 we obtain some examples of Gromov hyperbolic spaces in almost complex manifolds :

Corollary 0.3. *Let (V, ω) be a symplectic manifold and let D be a smooth relatively compact domain contained in V . Assume that ∂D is connected and of contact type. Then there exists an almost complex structure J on V , tamed by ω , for which the following equivalence is satisfied :*

$(D, d_{(D, J)})$ is Gromov hyperbolic if and only if D does not contain Brody J -holomorphic curves.

Corollary 0.4. *Let $D = \{\rho < 0\}$ be a smooth relatively compact domain in an almost complex manifold (M, J) , where ρ is a smooth defining function of D , strictly J -plurisubharmonic in a neighborhood of the closure \overline{D} of D . Then $(D, d_{(D, J)})$ is Gromov hyperbolic.*

In the next Section we present the necessary preliminaries. In Section 2, we present the proof of Theorem 0.1 and of Corollary 0.3. In Section 3 we give the proof of Corollary 0.4.

Acknowledgments. The authors would like to thank G.Della Sala for helpful discussions on the Morse theory.

1. PRELIMINARIES

1.1. Almost complex manifolds and pseudoholomorphic discs. An *almost complex structure* J on a real smooth manifold M is a $(1, 1)$ tensor field which satisfies $J^2 = -Id$. We suppose that J is smooth. The pair (M, J) is called an *almost complex manifold*. We denote by J_{st} the standard integrable structure on \mathbb{C}^n for every n . A differentiable map $f : (M', J') \rightarrow (M, J)$ between two almost complex manifolds is said to be (J', J) -*holomorphic* if $J(f(p)) \circ d_p f = d_p f \circ J'(p)$, for every $p \in M'$. In case $M' = \Delta$ is the unit disc in \mathbb{C} , such a map is called a *pseudoholomorphic disc*.

1.2. Strictly J -pseudoconvex domains. Let ρ be a smooth real valued function on a smooth almost complex manifold (M, J) . We denote by $d_J^c \rho$ the differential form defined by $d_J^c \rho(v) := -d\rho(Jv)$, where v is a section of TM . The *Levi form* of ρ at a point $p \in M$ and a vector $v \in T_p M$ is defined by $\mathcal{L}_J \rho(p, v) := dd_J^c \rho(p)(v, J(p)v)$. We say that ρ is *strictly J -plurisubharmonic* if $\mathcal{L}_J \rho(p, v) > 0$ for any

$p \in M$ and $v \neq 0 \in T_p M$. The boundary of a domain D is *strictly J -pseudoconvex* if at any point $p \in \partial D$ there exists a smooth strictly J -plurisubharmonic function ρ defined in a neighborhood U of p in M satisfying $\nabla \rho \neq 0$ on $\partial D \cap U$ and such that $D \cap U = \{\rho < 0\}$. We say that a domain $D = \{\rho < 0\}$ is a *strictly J -pseudoconvex region* in (M, J) if ρ is a smooth defining function of D , strictly J -plurisubharmonic in a neighborhood of \overline{D} .

1.3. Hypersurfaces of contact type. Let (V, ω) be a symplectic manifold, namely ω is a closed, non-degenerate, skew symmetric 2-form on the smooth manifold V . Let Γ be a hypersurface contained in V . We say that Γ is of *contact type* if there is a vector field X , defined near Γ , transverse to Γ and pointing outwards, such that $d(i(X)\omega) = \omega$. The 1-form $\alpha = i(X)\omega$ is a *contact form* on Γ and it defines a *contact structure* $\zeta = \ker \alpha$ on Γ .

1.4. The Kobayashi pseudometric. The existence of local pseudoholomorphic discs proved in [27] allows to define the *Kobayashi pseudometric* $K_{(M,J)}$ for $p \in M$ and $v \in T_p M$:

$$K_{(M,J)}(p, v) := \inf \left\{ \frac{1}{r} > 0, u : \Delta \rightarrow (M, J) \text{ } J\text{-holomorphic}, u(0) = p, d_0 u(\partial/\partial x) = rv \right\},$$

and its integrated pseudodistance $d_{(M,J)}$:

$$d_{(M,J)}(p, q) := \inf \left\{ \int_0^1 K_{(M,J)}(\gamma(t), \gamma'(t)) dt, \gamma : [0, 1] \rightarrow M, \gamma(0) = p, \gamma(1) = q \right\},$$

for $p, q \in M$. The manifold (M, J) is *Kobayashi hyperbolic* if $d_{(M,J)}$ is a distance. Notice that in case $D = \{\rho < 0\}$ is a strictly J -pseudoconvex region in (M, J) , there are no Brody J -holomorphic curves, namely nonconstant J -holomorphic lines $u : \mathbb{C} \rightarrow D$, contained in D and thus D is Kobayashi hyperbolic.

1.5. Gromov hyperbolic spaces. In this section we give some backgrounds about Gromov hyperbolic spaces. Let (X, d) be a metric space. The *Gromov product* of two points $x, y \in X$ with respect to a base point $\omega \in X$ is defined by $(x|y)_\omega := \frac{1}{2}(d(x, \omega) + d(y, \omega) - d(x, y))$. The Gromov product measures the failure of the triangle inequality to be an equality and is always nonnegative. The metric space X is *Gromov hyperbolic* if there is a nonnegative constant δ such that for any $x, y, z, \omega \in X$ one has:

$$(1.1) \quad (x|y)_\omega \geq \min((x|z)_\omega, (z|y)_\omega) - \delta.$$

We point out that (1.1) can be also written as follows:

$$(1.2) \quad d(x, y) + d(z, \omega) \leq \max(d(x, z) + d(y, \omega), d(x, \omega) + d(y, z)) + 2\delta,$$

for $x, y, z, \omega \in X$.

There is a family of metric spaces for which Gromov hyperbolicity may be defined by means of geodesic triangles. A metric space (X, d) is said to be *geodesic space* if any two distinct points $x, y \in X$ can be joined by a *geodesic segment*, that is the image of an isometry $\gamma : [0, d(x, y)] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. Such a segment is denoted by $[x, y]$. A *geodesic triangle* in X is a subset $[x, y] \cup [y, z] \cup [z, x]$ for some $x, y, z \in X$. For a geodesic space (X, d) , one may define equivalently (see [16]) the Gromov hyperbolicity in term of geodesic triangles. The geodesic space X is *Gromov hyperbolic* if there is a nonnegative constant δ such that for any geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ and any $\omega \in [x, y]$ one has $d(\omega, [y, z] \cup [z, x]) \leq \delta$.

The canonical morphisms of Gromov hyperbolic spaces are the following:

Definition 1.1. Let $f : (X, d) \rightarrow (X', d')$ be a map between two metric spaces. We say that

- (1) f is a rough isometry if there exist a positive constant $c > 0$ such that for every $x, y \in X$:

$$d(x, y) - c \leq d'(f(x), f(y)) \leq d(x, y) + c.$$

(2) f is a quasi-isometry if there exist two positive constants $\lambda, c > 0$ such that for every $x, y \in X$:

$$\frac{1}{\lambda}d(x, y) - c \leq d'(f(x), f(y)) \leq \lambda d(x, y) + c.$$

Naturally, we say that two spaces are *roughly isometric* (resp. *quasi-isometric*) if there exists a rough isometry (resp. quasi-isometry) between them. It can be easily shown that if (X, d) and (X', d') are roughly isometric then (X, d) is Gromov hyperbolic if and only if (X', d') is. In case we consider quasi-isometric spaces, the spaces need to be both geodesic. This is provided by the following theorem (see Theorem 12 p. 88 [16]) which is crucial for us:

Theorem 1.2. *Let (X, d) and (X', d') be two quasi-isometric geodesic metric spaces. Then (X, d) is Gromov hyperbolic if and only if (X', d') is.*

2. PROOF OF THEOREM 0.1 AND OF COROLLARY 0.3

2.1. Proof of Theorem 0.1. We first notice that the Gromov hyperbolicity of $(D, d_{(D, J)})$ implies that $d_{(D, J)}$ is a metric on D , meaning that (D, J) is Kobayashi hyperbolic. In particular, D does not contain any Brody J -holomorphic curve.

Conversely if D does not contain any Brody J -holomorphic curve then (D, J) is Kobayashi hyperbolic. Indeed, assuming by contradiction that (D, J) is not Kobayashi hyperbolic we may construct on D a sequence of J -holomorphic discs with derivatives exploding at the centers. Since ∂D is strictly J -pseudoconvex, it follows from estimates of the Kobayashi infinitesimal pseudometric (see [15, 20]) that such discs do not approach the boundary ∂D of D . Hence by a renormalization process we construct a Brody curve contained in D , which is a contradiction.

The *complex tangent space* of ∂D is by definition $T^J \partial D := T \partial D \cap J \partial D$. Since ∂D is strictly J -pseudoconvex, the complex tangent space $T^J \partial D$ is a contact structure. More precisely, let ρ be a strictly J -plurisubharmonic function defined in a neighborhood U of ∂D and such that $\partial D = \{\rho = 0\}$, $D \cap U = \{\rho < 0\}$. Consider the one-form $d_J^c \rho$ and let α be its restriction to the tangent bundle $T \partial D$. It follows that $T^J \partial D = \text{Ker} \alpha$. Due to the strict J -plurisubharmonicity of ρ , the two-form $\omega := dd_J^c \rho$ is a symplectic form on U that tames J , $T^J \partial D$ is a contact structure and α is a contact form for $T^J \partial D$. Consequently and since the boundary ∂D of D is connected, a theorem due to Chow [10] states that any two points in ∂D may be connected by a C^1 *horizontal curve*, i.e. a curve $\gamma : [0, 1] \rightarrow \partial D$ satisfying $\gamma'(s) \in T_{\gamma(s)}^J \partial D$ for every $s \in [0, 1]$. This allows to define the *Carnot-Carathéodory metric* as follows (see [3, 19]) :

$$d_H(p, q) := \{l_{\mathcal{L}_J \rho}(\gamma), \gamma : [0, 1] \rightarrow \partial D \text{ horizontal}, \gamma(0) = p, \gamma(1) = q\},$$

where $l_{\mathcal{L}_J \rho}(\gamma)$ is the *Levi length* of the horizontal curve γ defined by $l_{\mathcal{L}_J \rho}(\gamma) := \int_0^1 \mathcal{L}_J \rho(\gamma(s), \gamma'(s))^{\frac{1}{2}} ds$.

In the Euclidean case (\mathbb{R}^{2n}, J) , the natural idea of Z.M.Balogh and M.Bonk to prove the Gromov hyperbolicity of D endowed with the Kobayashi distance $d_{(D, J_{st})}$ is to construct a metric on D , based on the Carnot-Carathéodory metric, which satisfies (1.2) and which is quasi-isometric to $d_{(D, J_{st})}$ (see [1], [2]).

We first equip M with an arbitrary smooth Riemannian metric and we denote by dist the associated distance. For $p \in D$ we define a boundary projection (multivalued) map $\pi : D \rightarrow \partial D$ by $\delta(p) = \text{dist}(p, \pi(p)) = \text{dist}(p, \partial D)$. Notice that the map π is uniquely determined near the boundary. Set $N_\varepsilon(\partial D) := \{q \in D, \delta(q) \leq \varepsilon\}$ where ε is such that π is uniquely defined on $N_\varepsilon(\partial D)$, and define the *height* of p by $h(p) := \sqrt{\delta(p)}$. Then we define a metric $g : D \times D \rightarrow [0, +\infty)$ by:

$$g(p, q) := 2 \log \left(\frac{d_H(\pi(p), \pi(q)) + \max\{h(p), h(q)\}}{\sqrt{h(p)h(q)}} \right),$$

for $p, q \in D$ (see [2]). It is important to notice that different choices of a Riemannian metric and of a projection π give a different metric that coincides with g up to an additive constant. Since we are dealing with rough and quasi-isometries, that will not disturb our results.

It was proved by Z.M.Balogh and M.Bonk [2] that the metric g satisfies (1.2) and thus that the metric space (D, g) is Gromov hyperbolic. However the space (D, g) is not geodesic. In order to construct a geodesic Gromov hyperbolic metric space, we need to perturb the metric g . Following [6] we construct a new metric d as follows. For p (resp. q) in D let p_ε (resp. q_ε) be the point on the fiber $\pi^{-1}(\pi(p))$ (resp. $\pi^{-1}(\pi(q))$) with height $\sqrt{\varepsilon}$ and let $l_g(\gamma) := \sup_{0=t_0 < t_1 < \dots < t_n=1} \sum_{i=1}^n g(\gamma(t_{i-1}), \gamma(t_i))$. Then we define $d(p, q)$ by :

$$d(p, q) := \begin{cases} \inf\{l_g(\gamma), \gamma : [0, 1] \rightarrow N_\varepsilon(\partial D) \text{ smooth curve joining } p \text{ and } q\} & \text{for } p, q \in N_\varepsilon(\partial D), \\ d(p, q_\varepsilon) + \text{dist}(q, q_\varepsilon) & \text{for } p \in N_\varepsilon(\partial D), q \in D \setminus N_\varepsilon(\partial D), \\ \text{dist}(p, q) & \text{for } p, q \in D \setminus N_\varepsilon(\partial D) \text{ such that } \pi(p) = \pi(q), \\ \text{dist}(p, p_\varepsilon) + d(p_\varepsilon, q_\varepsilon) + \text{dist}(q, q_\varepsilon) & \text{for } p, q \in D \setminus N_\varepsilon(\partial D) \text{ such that } \pi(p) \neq \pi(q). \end{cases}$$

In case $(M, J) = (\mathbb{R}^{2n}, J)$, L.Blanc-Centi [6] proved that the metric space (D, d) is roughly isometric to (D, g) and thus Gromov hyperbolic and that moreover (D, d) is geodesic. The arguments used in [6] remain valid for domains in an almost complex manifold. Thus

Proposition 2.1. *The metric space (D, d) is geodesic and Gromov hyperbolic.*

The proof follows essentially the proof given by L.Blanc-Centi in [6]. For sake of completeness, we include the key points of the proof.

Proof of Proposition 2.1. We first notice that (D, g) and (D, d) are roughly isometric. It is sufficient to prove that these spaces are roughly isometric near the boundary, namely that there exists a positive constant $C > 0$ such that $g(p, q) - c \leq d(p, q) \leq g(p, q) + c$ for all $p, q \in N_\varepsilon(\partial D)$. This is obtained by considering different positions of p, q and by studying normal and horizontal curves. Normal curves are purely local objects since points considered belong to $N_\varepsilon(\partial D)$ and have the same boundary projection (see Lemma 1 in [6]). Horizontal curves joining two points $p, q \in N_\varepsilon(\partial D)$ with same height $h(p) = h(q)$ are defined as follows: since $(\partial D, d_H)$ is geodesic, we consider a geodesic curve α in ∂D joining $\pi(p)$ and $\pi(q)$. For each t , we consider the point $\gamma(t) \in N_\varepsilon(\partial D)$ in the fiber $\pi^{-1}(\alpha(t))$ with height $h(p) = h(q)$. Then γ defines a smooth horizontal curve in $N_\varepsilon(\partial D)$. These two kinds of curves being defined for manifolds the proof that (D, g) and (D, d) are roughly isometric is then given by [6]. And since (D, g) is Gromov hyperbolic, we obtain the Gromov hyperbolicity of the metric space (D, d) .

The proof that (D, d) is geodesic is achieved by studying different positions of two points $p, q \in D$. However, according to the definition of the metric d , it reduces to two cases: $p, q \in D \setminus N_\varepsilon(\partial D)$ satisfying $\pi(p) = \pi(q)$ and $p, q \in N_\varepsilon(\partial D)$. If $p, q \in D \setminus N_\varepsilon(\partial D)$ and satisfy $\pi(p) = \pi(q)$, then d coincides with the metric dist induced by a Riemannian metric. Since $(\overline{D \setminus N_\varepsilon(\partial D)}, \text{dist})$ is compact, it is complete and thus according to the Hopf-Rinow Theorem, it is geodesic. In case $p, q \in N_\varepsilon(\partial D)$, the existence of a geodesic curve provided by [6] in the Euclidean case (\mathbb{R}^{2n}, J) (see Lemma 4 [6]) remains valid in the case of any almost complex manifold. \square

Remark 2.2. *It follows from the construction of the metric g , involving the Carnot-Carathéodory metric d_H , that the boundary $\partial_G D$ of (D, g) as a Gromov hyperbolic space, and equivalently of (D, d) , can be identified with the boundary ∂D of the domain D .*

The space (D, d) being geodesic and Gromov hyperbolic, it remains to show that the metric space $(D, d_{(D,J)})$ is geodesic and quasi-isometric to (D, d) . According to Z.M.Balogh and M.Bonk [2], this reduces to considering estimates for the Kobayashi metric $K_{(D,J)}$ near the boundary of D . In the case of almost complex manifolds, we will use the following precise estimates of the Kobayashi metric obtained in [15] :

Theorem 2.3. *let $D' = \{\rho < 0\}$ be a relatively compact strictly J -pseudoconvex region in an almost complex manifold (M, J) . Then there exists a positive constant C such that :*

(2.1)

$$\frac{1}{C} \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{\|v\|^2}{|\rho(p)|} \right]^{1/2} \leq K_{(D',J)}(p, v) \leq C \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{\|v\|^2}{|\rho(p)|} \right]^{1/2}$$

for every $p \in D'$ and every $v \in T_p M$.

As a classical consequence of the lower estimate of the Kobayashi metric in (2.1), we obtain the completeness of the metric space $(D, d_{(D,J)})$, which implies, by the Hopf-Rinow Theorem, that $(D, d_{(D,J)})$ is geodesic. Based on sharp estimates provided by D.Ma [24], Z.M.Balogh and M.Bonk [2] proved that the Kobayashi distance of a bounded strictly J_{st} -pseudoconvex domain in \mathbb{R}^{2n} is roughly similar to the metric g . It is important to notice that their proof is purely metric and does not use any complex geometry or complex analysis arguments. In particular, following their proof and replacing the estimates provided by [24] by the estimates (2.1), we obtain that $(D, d_{(D,J)})$ is quasi-isometric to (D, d) . Finally, Theorem 1.2 implies that the space $(D, d_{(D,J)})$ is Gromov hyperbolic which proves Theorem 0.1. \square

Remark 2.4. *The connectedness of ∂D in Theorem 0.1 is necessary for our approach. If ∂D is not connected one may define the metric g by choosing arbitrarily one of the components of ∂D and repeating the construction of Subsection 2.1. However the induced distance g is not equivalent to $d_{(D,J)}$. It would be natural to consider the case of domains with non connected boundaries, such as the example provided by D.McDuff [25].*

2.2. Proof of Corollary 0.3. Since $\omega|_\zeta$ is nondegenerate, there exists an almost complex structure J defined on a neighborhood of ∂D and tamed by ω , such that ζ is J -invariant and $d\alpha(v, Jv) > 0$ for all non-zero $v \in \zeta$. Let $\rho : V \rightarrow \mathbb{R}$ is a smooth function such that $\partial D = \rho^{-1}(0)$ and $d\rho(X) > 0$ where X is given by Subsection 1.3. Then for $v \in \zeta$ we have $Jv \in \zeta$ and $\text{Ker} d_J \rho|_{\partial D} = \text{Ker} \alpha|_{\partial D}$. This implies that there is a positive function μ defined near ∂D such that $dd_J^c \rho = \mu d\alpha$ on ζ (see [25] Lemma 2.4). In particular ∂D is strictly J -pseudoconvex. Since ∂D is connected, we may apply Theorem 0.1 to conclude the proof of Corollary 0.3.

3. RELATIVELY COMPACT STRICTLY J -PSEUDOCONVEX REGIONS

We focus now on the proof of Corollary 0.4. Applying Theorem 0.1 this reduces to proving the connectedness of ∂D .

Lemma 3.1. *The boundary ∂D of a relatively compact strictly J -pseudoconvex region is connected.*

The proof of Lemma 3.1 is based on the Morse theory and more precisely on the fact that, considering a domain defined by a Morse function, namely a smooth function which Hessian is non-degenerate at its critical points, a change in its topology can only occur at critical points (see [26] for more details). We first need the following version of the Morse Lemma:

Lemma 3.2. *Let (M, J) be an almost complex manifold. Let ρ be a strictly J -plurisubharmonic Morse function on M . Then there exists local coordinates such that at each critical point one has*

$$(3.1) \quad \rho(z) = \rho(0) + \sum_{j=1}^n x_j^2 + \sum_{j=1}^n a_j y_j^2 + O(|z|^3)$$

where $a_j = \pm 1$. In particular the Morse index of a critical point is smaller than n .

Proof of Lemma 3.2. Let p be a critical point of ρ . We use a normalization due to K.Diederich and A.Sukhov (see Lemma 3.2 and Proposition 3.5 in [13]). In the associated coordinate system centered at $p = 0$, the Levi forms of ρ at the origin with respect to J_{st} and J coincide. In particular ρ is strictly J_{st} -plurisubharmonic at the origin. Then

$$\rho(z) = \rho(0) + \sum_{i,j=1}^n a_{ij} z_i \bar{z}_j + \Re \sum_{i,j=1}^n b_{ij} z_i z_j + O(|z|^3),$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are respectively Hermitian and symmetric matrices. Applying a linear transformation, we can reduce A to the identity $A = I_n$. Moreover we can make a unitary transformation $z \mapsto Uz$ preserving $\sum_{j=1}^n |z_j|^2$ and changing the matrix B into a diagonal matrix $U^t B U$ with nonnegative elements (see Lemma 7.2 of [12]). Then the expression of ρ reduces to

$$\rho(z) = \rho(0) + \sum_{j=1}^n |z_j|^2 + \sum_{j=1}^n \alpha_j \Re z_j^2 + O(|z|^3),$$

where $\alpha_j \geq 0$ for $j = 1, \dots, n$. This gives (3.1). \square

Proof of Lemma 3.1. After a small perturbation of ρ that does not change the strict J -plurisubharmonicity, we can assume that ρ is a Morse function and that, moreover, every critical set contains only one critical point. It follows that ρ has a finite number of critical points. Now, assume that the boundary $\partial D = \{\rho = 0\}$ of D has $m \geq 2$ connected components. The main point of the Morse theory is that the topology of $D \cap \{\rho \geq -t\}$ for $t \geq 0$ changes only at a critical level set. Denote by $-c_1 \geq -c_2 \geq \dots \geq -c_r$ the real numbers for which critical level sets of ρ occur, and denote by $p_i \in \{\rho = -c_i\}$, $i = 1, \dots, r$, the critical points of ρ . According to Lemma 3.2, the Morse index k_i of p_i satisfies $0 \leq k_i \leq n$ for $i = 1, \dots, r-1$ and since the domain D is relatively compact then $k_r = 0$. We need to describe locally the topology of D in a neighborhood of a critical level set.

Let p_i be a critical point of Morse index $2 \leq k_i \leq n$. In a coordinate system centered at p_i given by Lemma 3.2, ρ can be written

$$\rho(z) = -c_i + \sum_{j=1}^n x_j^2 + \sum_{j=1}^n a_j y_j^2 + O(|z|^3)$$

with at least two negative coefficients, say a_{j_1} and a_{j_2} . Since $\{\rho = -c_i + \varepsilon\} \cap \{y_1 = y_2 = \dots = y_n = 0\}$ is a perturbation of a n -sphere, it follows that locally the level set $\{\rho = -c_i + \varepsilon\}$ for small positive $\varepsilon > 0$ is connected. Moreover in the plane $\text{span}\{y_{j_1}, y_{j_2}\}$, the level set $\{\rho = -c_i - \varepsilon\}$ is a perturbation of a circle, and thus locally $\{\rho = -c_i - \varepsilon\}$ for small positive $\varepsilon > 0$ is connected.

Let p_i be a critical point of ρ of Morse index 1. It follows that in a coordinate system centered at p_i provided by Lemma 3.2 ρ has the form

$$\rho(z) = -c_i + \sum_{j=1}^n x_j^2 + \sum_{j \neq j_0} y_j^2 - y_{j_0}^2 + O(|z|^3)$$

for some j_0 . Locally the level set $\{\rho = -c_i + \varepsilon\}$ is connected whereas $\{\rho = -c_i - \varepsilon\}$ is not.

Finally, in a coordinate system centered at p_i of Morse index 0 provided by Lemma 3.2 ρ has the form

$$\rho(z) = -c_i + \sum_{j=1}^n x_j^2 + \sum_{j=1}^n y_j^2 + O(|z|^3),$$

and so locally the level set $\{\rho = -c_i + \varepsilon\}$ is connected and $\{\rho = -c_i - \varepsilon\}$ is empty.

This implies that starting with $D \cap \{\rho \geq -\varepsilon\}$ with $m \geq 2$ connected components for $\varepsilon > 0$ small enough, the number of connected components of $D \cap \{\rho \geq -t\}$ cannot decrease when $t \rightarrow -c_r$, contradicting the fact that level sets nearby $\{\rho = -c_r\}$ are connected. □

REFERENCES

- [1] Balogh, Z.M., Bonk, M. *Pseudoconvexity and Gromov hyperbolicity*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 597-602.
- [2] Balogh, Z.M., Bonk, M. *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. **75** (2000), 504-533.
- [3] Bellaïche, A. *The tangent space in sub-Riemannian geometry*, Sub-Riemannian geometry, Progr. Math. **144**, Birkhäuser, Basel (1996), 1-78.
- [4] Bertrand, F. *Sharp estimates of the Kobayashi metric and Gromov hyperbolicity*, J. Math. Anal. Appl. **345** (2008), 825-844.
- [5] Biolley, A.-L. *Floer homology, symplectic and complex hyperbolicities*, ArXiv: math. SG/0404551.
- [6] Blanc-Centi, L. *On the Gromov hyperbolicity of the Kobayashi metric on strictly pseudoconvex regions in the almost complex case*, Math. Z. **263** (2009), 481-498.
- [7] Bonk, M.; Heinonen, J.; Koskela, P. *Uniformizing Gromov hyperbolic spaces*. Astérisque **270** (2001), viii+99 pp.
- [8] Bonk, M., Schramm, O. *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), 266-306.
- [9] Byun, J., Gaussier, H., Lee, K.H. *On the automorphism group of strongly pseudoconvex domains in almost complex manifolds*, Ann. Inst. Fourier **59** (2009), 291-310.
- [10] Chow, W.L. *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117** (1939), 98-105.
- [11] Coornaert, M.; Delzant, T.; Papadopoulos, A. *Géométrie et théorie des groupes*. (French) [Geometry and theory of groups] Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups] With an English summary. Lecture Notes in Mathematics, **1441**. Springer-Verlag, Berlin, 1990. x+165 pp.
- [12] Coupet, B., Sukhov, A., Tumanov, A. *Proper J -holomorphic discs in Stein domains of dimension 2*, Amer. J. Math. **131** (2009), 653-674.
- [13] Diederich, K., Sukhov, A. *Plurisubharmonic exhaustion functions and almost complex Stein structures*, Michigan Math. J. **56** (2008), 331-355.
- [14] Forstneric, F., Rosay, J.-P. *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. **279** (1987), 239-252.
- [15] Gaussier, H., Sukhov, A. *Estimates of the Kobayashi metric on almost complex manifolds*, Bull. Soc. Math. France **133** (2005), 259-273.
- [16] Ghys, E., de la Harpe, P. (Eds.) *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. **83**, Birkhäuser Boston, Boston, 1990.
- [17] Graham, I. *Boundary behaviour of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in \mathbb{C}^n with smooth boundary*, Trans. Amer. Math. Soc. **207** (1975), 219-240.
- [18] Gromov, M. *Hyperbolic groups*, in "Essays in group theory" (G. Gernsten, ed.), Math. Sci. Res. Inst. Publ. Springer (1987), 75-263.
- [19] Gromov, M. *Carnot-Carathéodory spaces seen from within*, Sub-Riemannian geometry, Progr. Math. **144**, Birkhäuser, Basel, 1996, 79-323.
- [20] Ivashkovich, S., Rosay, J.-P. *Schwarz-type lemmas for solutions of $\bar{\partial}$ -inequalities and complete hyperbolicity of almost complex manifolds*, Ann. Inst. Fourier **54** (2004), 2387-2435.
- [21] Kobayashi, S. *negative vector bundles and complex Finsler structures*, Nagoya Math. J. **57** (1975), 153-166.
- [22] Kobayashi, S. *Hyperbolic complex spaces*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **318**. Springer-Verlag, Berlin, 1998. xiv+471 pp.
- [23] Lempert, L. *La métrique de Kobayashi et la représentation des domaines sur la boule* (French. English summary) [The Kobayashi metric and the representation of domains on the ball], Bull. Soc. Math. France **109** (1981), 427-474.
- [24] Ma, D. *Sharp estimates of the Kobayashi metric near strongly pseudoconvex points*, The Madison Symposium on Complex Analysis (Madison, WI, 1991), Contemp. Math. **137**, Amer. Math. Soc., Providence, RI, 1992, 329-338.

- [25] McDuff, D. *Symplectic manifolds with contact type boundaries*, Invent. Math. **103** (1991), 651-671.
- [26] Milnor, J. *Morse theory*, Annals of Mathematics Studies **51**, Princeton University Press.
- [27] Nijenhuis, A., Wolf, W. *Some integration problems in almost-complex and complex manifolds*, Ann. Math. **77** (1963), 429-484.

Florian Bertrand

Department of Mathematics, University of Vienna

Nordbergstrasse 15,

Vienna, 1090, Austria

E-mail address : florian.bertrand@univie.ac.at

Hervé Gaussier

(1) UJF-Grenoble 1, Institut Fourier, Grenoble, F-38402, France

(2) CNRS UMR5582, Institut Fourier, Grenoble, F-38041, France

E-mail address : herve.gaussier@ujf-grenoble.fr